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# ON THE APPLICATION OF SPINORS TO THE PROBLEMS OF CELESTIAL MECHANICS

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TO THE PROBLEMS OF CELESTIAL MECHANICS

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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## ABSTRACT

Since Newton, the stumbling-block in celestial mechanics has been the three-body problem. Only restricted cases have yielded solutions. This paper describes a new tool for solving the perturbation problem—"spinors," a vectorial concept (in the complex plane) originally used in atomic physics but now applied to celestial mechanics. The two-body problem is discussed and two devices are introduced: "pseudo-time" and the Levi-Civita transformations. Then spinors are shown to make possible the extension of these transformations to three dimensions. Perturbation problems are discussed; notably a body moving around the sun and being acted on by the orbital motion of a major planet.

## CONTENTS

|   |    |
|---|----|
| Abstract . . . . .  | ii |
| REGULARIZATION OF THE KEPLER PROBLEM—<br>TRANSFORMATIONS OF LEVI-CIVITA . . . . . | 1  |
| LEVI-CIVITA TRANSFORMATIONS IN SPACE—<br>INTRODUCTION OF SPINORS . . . . .        | 10 |
| THE DIFFERENTIAL EQUATIONS OF THE<br>SPINOR OF A PERTURBED KEPLER MOTION . . .    | 18 |
| VECTORIAL REPRESENTATION OF SPINORS . . . .                                       | 24 |
| GENERAL CONSIDERATIONS ON REGULARIZA-<br>TION IN DYNAMICAL PROBLEMS . . . . .     | 32 |
| Acknowledgment . . . . .  | 41 |
| References . . . . .  | 41 |

# ON THE APPLICATION OF SPINORS TO THE PROBLEMS OF CELESTIAL MECHANICS\*

by

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## REGULARIZATION OF THE KEPLER PROBLEM— TRANSFORMATIONS OF LEVI-CIVITA

Application of spinors to the problems of celestial mechanics was first made by P. Kustaanheimo and E. Stiefel. Originally, spinors were the creation of atomic physicists and mathematicians; now astronomers have taken them up. Spinors are symbols by which spin (the rotation of an elementary particle) can be mathematically described. In celestial mechanics spinors are no longer connected to their conceptual origins; they have become a formal aid in simplifying theory and its application to special problems.

Spinors may be defined in different ways. One definition identifies a spinor with an arrow in the complex plane leading from a complex number  $z_1$  to another complex number  $z_2$ . Mathematically we may represent a spinor by the matrix

$$S = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix},$$

composed of the four real numbers  $u_i, v_i$ , where the complex numbers are  $z_1 = u_1 + iv_1$  and  $z_2 = u_2 + iv_2$ . Or we may consider the spinor  $S$  as a four-dimensional vector

$$S = \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix}$$

\*This essay originates from lectures given at the Goddard Space Flight Center, Laboratory for Theoretical Studies, in November, 1966.

Mathematicians have elaborated a complete algebra of spinors (Reference 1), closely related to the algebra of matrices or vectors. It would take too long and lead too far from the main object of this paper to give even a short introduction to spinor algebra. However, we may use instead of spinors the corresponding matrices or vectors or pairs of complex numbers, the algebraic properties of which are well known.

Our object is to solve the equations of motion of celestial-bodies. This means (1) the undisturbed motion (in conic sections) of planets, comets, or artificial satellites around their central body, and (2) their motion when perturbed by the attraction of other bodies of the solar system. The differential equations of motion and their solutions in classical form have unsatisfactory features: lack of symmetry, lack of universality, and the existence of singularities. Spinors or their vectorial analogies are found very helpful in removing these difficulties.

Consider, first, the simple case of the two-body problem in the Cartesian  $xy$ -plane. There, symmetrization, regularization, and generalization can be performed in an elementary manner by convenient transformations of the coordinates and the independent variable. (Spinors are not yet needed; they will be introduced later, for studying the motion of celestial bodies in space.)

The equations for the undisturbed Keplerian motion of a mass point  $m_2$  around a central mass  $m_1$  in the  $xy$ -plane are:

$$\ddot{x} + \frac{x}{r^3} = 0, \quad \ddot{y} + \frac{y}{r^3} = 0, \quad r = \sqrt{x^2 + y^2}, \quad (1)$$

provided that the dimensions of time, mass and length are chosen so that  $k^2 (m_1 + m_2) = 1$ . The general solution can be simply obtained if we use (instead of time  $t$ ) the eccentric anomaly  $E$  as an independent variable. Let

$$x = r \cos(v - \omega), \quad y = r \sin(v - \omega),$$

where  $r$  and  $v$  are polar coordinates and  $\omega$  is an arbitrary constant. Furthermore

$$r \cos v = a(\cos E - e), \quad r \sin v = a\sqrt{1 - e^2} \sin E,$$

where  $a$  and  $e$  are two other arbitrary constants. Finally,  $E$  may be calculated from Kepler's equation

$$E - e \sin E = a^{-3/2} (t - T).$$

The time of perihelion passage  $T$  is the last of the four independent integration constants required. The equation of the trajectory in the  $r, E$  plane is

$$r = a(1 - e \cos E).$$

This is the equation of an ellipse with one focus at the origin. This simple solution serves only for elliptic orbits where eccentricity  $e < 1$ . For hyperbolic orbits ( $e > 1$ ),  $E$  is an imaginary angle and we must change the algorithm for calculating the coordinates as functions of the time. In the case of parabolic orbits ( $e = 1$ ,  $a = \infty$ ), the formulas lose their meaning; in fact, numerical calculations become intricate for orbits with eccentricities close to unity. These various difficulties have been mastered, in classical celestial mechanics, by using different methods of calculation for different types of orbits.

The lack of universality can be overcome by using for the new independent variable the "pseudo-time,"  $s = \sqrt{a}(E - E_0)$ , where  $E_0$  is the value of  $E$  at an arbitrary initial time  $t_0$ . Then

$$\frac{ds}{dt} = \sqrt{a} \frac{dE}{dt} = \frac{1}{r}$$

is positive and real for all types of orbits, and  $s$  itself is zero for  $t = t_0$ . In the parabolic case, where  $a = \infty$  and  $E = 0$ ,

$$s = \int_{t_0}^t \frac{dt}{r}$$

remains finite; the discontinuity at  $e = 1$  disappears if  $s$  is used as independent variable instead of  $E$  or  $t$ . The coordinates  $x$ ,  $y$  of a moving body can be expressed as functions of  $s$  in forms that do not depend on the type of orbit described.

This transformation also solves the problem of regularization. The energy-integral is given by

$$v^2 = \dot{x}^2 + \dot{y}^2 = \frac{2}{r} - \frac{1}{a}, \quad (2)$$

where  $v$  is the velocity and  $a$  is the major semi-axis of the conic section along which the planet is moving. This equation shows that in the case of a collision between the two bodies,  $v$  approaches infinity as  $r$  approaches zero. Equations 1 show that the acceleration approaches infinity. Numerical computation of the orbital motion is difficult if the body, at its pericenter, approaches the central mass very closely. Introducing  $s$  instead of  $t$  as independent variable eliminates this singularity. If primes denote differentiation with respect to  $s$ , then, for any function  $q(t)$ ,

$$\dot{q} = q' \frac{ds}{dt} = \frac{1}{r} q'.$$

Therefore

$$v^2 = \frac{x'^2 + y'^2}{r^2} = \frac{2}{r} - \frac{1}{a}.$$

$$x'^2 + y'^2 = 2r - \frac{r^2}{a} . \quad (3)$$

The velocity, expressed with respect to pseudo-time  $s$ , (given by the square root of the left side of Equation 3) approaches zero as  $r$  approaches zero. The formulas giving  $x(s)$ ,  $y(s)$ , and  $z(s)$  in a spatial  $xyz$ -system are not only valid for all types of orbits, but they are not affected by singularities.

Pseudo-time  $s$  may be derived as a function of  $\tau = t - t_0$  from the differential relation

$$d\tau = r ds .$$

The integral of this equation is analogous to Kepler's equation; it is likewise valid for all types of orbits.

This is more apparent if the equations of two-body motion are written with  $s$  as an independent variable. Then, from  $\dot{x} = x'/r$ , second differentiation gives:

$$\ddot{x} = -\frac{x}{r^3} = \frac{1}{r} \left( \frac{x''}{r} - \frac{x' r'}{r^2} \right),$$

i.e.,

$$x'' = \frac{x' r' - x}{r} , \quad y'' = \frac{y' r' - y}{r} . \quad (4)$$

These equations must be completed by differential relations for  $r(s)$ . From

$$r^2 = x^2 + y^2 ,$$

differentiating twice gives

$$r'^2 + rr'' = x'^2 + y'^2 + xx'' + yy'' = x'^2 + y'^2 + \frac{x}{r} (x' r' - x) + \frac{y}{r} (y' r' - y) ,$$

i.e.,

$$rr'' = x'^2 + y'^2 - r .$$

Putting in Equation 3 for the square of the velocity gives

$$r'' = 1 - \frac{r}{a} \quad \text{or} \quad \frac{1}{a} = \alpha^2 = \frac{1 - r''}{r} = \text{const.}$$



Therefore,  $r(s)$  is a solution of

$$r'' + \alpha^2 r = 1 \text{ or } r''' + \alpha^2 r' = 0.$$

Another differentiation of Equations 4 gives

$$rx''' + r' x'' = r' x'' + x' r'' - x',$$

i.e.,

$$rx''' = x' (r'' - 1), \quad x''' + \frac{1-r''}{r} x' = 0, \quad x''' + \alpha^2 x = 0.$$

Therefore,  $x, y, z$ , and  $r$  satisfy, as functions of  $s$ , differential equations of the same type:

$$q''' + \alpha^2 q' = 0,$$

where  $\alpha^2$  is a real constant which is positive for ellipses, zero for parabolas, and negative for hyperbolas.

Integration of these linear differential equations is very easy. If  $q$  is the general symbol for  $r, x, y$ , or  $z$ , then: for

$$\alpha^2 > 0: \quad q = q_0 + q_1 \cos \alpha s + q_2 \sin \alpha s,$$

for

$$\alpha^2 = 0: \quad q = q_0 + q_1 s + q_2 s^2,$$

for

$$\alpha^2 < 0: \quad q = q_0 + q_1 \cosh \beta s + q_2 \sinh \beta s, \quad (\beta^2 = -\alpha^2 > 0).$$

These expressions are still distinct for different types of orbits. They receive a universal form from the introduction of the "c-functions" (Reference 5, Chapter 5). Let  $\lambda = \alpha s$ . Then  $\lambda^2$  is always real, and

$$c_0(\lambda^2) = \cos \lambda, \quad c_1(\lambda^2) = \frac{\sin \lambda}{\lambda}, \quad c_2(\lambda^2) = \frac{1 - \cos \lambda}{\lambda^2}, \quad c_3(\lambda^2) = \frac{\lambda - \sin \lambda}{\lambda^3} \quad (5)$$

are functions of  $\lambda^2$  and real for all types of orbits. In the parabolic case,  $\lambda$  is exactly zero, and the  $c$ -functions become constants,  $c_n = 1/n!$ . Between the  $c$ -functions there exist the relations

$$c_n = \frac{1}{n!} - \lambda^2 c_{n+2}, \quad \frac{d}{ds} (c_n s^n) = c_{n-1} s^{n-1}, \quad \int_0^s c_n s^n ds = c_{n+1} s^{n+1}. \quad (6)$$

From these relations it follows that the general solution of the differential equation

$$q''' + \alpha^2 q' = 0$$

may be written

$$q = q_0 + c_1 s q_0' + c_2 s^2 q_0'',$$

if  $q_0, q_0', q_0''$  are the values of  $q, q', q''$  for  $s = 0$ . Differentiating these expressions three times (regarding the definition of the  $c$  functions and the relations between them) shows that  $q''' = -\alpha^2 q'$ . Note that  $q_0$  and  $q_0'$  may be arbitrarily given as initial conditions. Then  $q_0''$  can be computed, in the case of  $r$  and  $x$ , from

$$r_0'' = 1 - \alpha^2 r_0, \quad x_0'' = \frac{x_0' r_0' - x_0}{r_0}.$$

To derive the time  $\tau = t - t_0$  as function of  $s$ , integrate

$$d\tau = r ds = (r_0 + c_1 s r_0' + c_2 s^2 r_0'') ds.$$

Using the integral relations of Equation 6 for the  $c$ -functions gives

$$\tau = \int_0^s r ds = s r_0 + c_2 s^2 r_0' + c_3 s^3 r_0''.$$

From this "main equation" of the two-body problem, which is analogous to Kepler's equation,  $s = s(\tau)$  may be derived by iteration.

The above solution of Kepler's problem is universal and free from singularities, but not unique. Levi-Civita has proposed another, simpler and possibly more suitable as an approximate solution when perturbations are present. Levi-Civita transforms the Cartesian coordinates  $x, y$  by

putting

$$x = \xi^2 - \eta^2, \quad y = 2\xi\eta, \quad r = \xi^2 + \eta^2. \quad (7)$$

The trajectory of the motion in the  $xy$ -plane (the positive  $x$ -axis being directed to the perihelion) is given by

$$x = r \cos v, \quad y = r \sin v, \quad r(1 + e \cos v) = a(1 - e^2)$$

and is transformed into

$$r + ex = (\xi^2 + \eta^2) + e(\xi^2 - \eta^2) = a(1 - e^2),$$

i.e.,

$$\frac{\xi^2}{a(1 - e)} + \frac{\eta^2}{a(1 + e)} = 1.$$

This is the central equation of an ellipse, the semi-axes of which are  $\sqrt{a(1 - e)}$  and  $\sqrt{a(1 + e)}$ , i.e. the square roots of the perihelion and aphelion distances of the true orbit (see Figure 1a). If, in the  $xy$ -plane the body moves around the sun in the direction APA, it moves in the  $\xi\eta$ -plane only along one half of the ellipse. In other words, one revolution in the  $\xi\eta$ -plane corresponds to two revolutions in the  $xy$ -plane. Indeed, if  $\xi$  and  $\eta$  are expressed by polar coordinates  $R, \chi$ :

$$\xi = R \cos \chi, \quad \eta = R \sin \chi,$$

From these equations and Equations 7, it can be shown that

$$\xi^2 = x + \frac{\sqrt{x^2 + y^2}}{2},$$

$$\eta^2 = -x + \frac{\sqrt{x^2 + y^2}}{2},$$

$$R = \sqrt{r},$$

$$\chi = \frac{v}{2}.$$

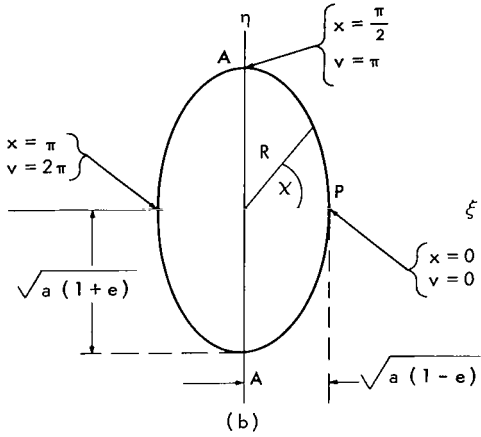
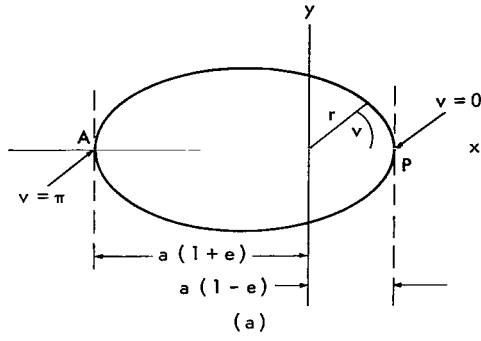


Figure 1—Trajectory in Cartesian and Levi-Civita coordinates.

See Figure 1b. Also:

$$R^2 (1 + e \cos 2X) = p = a(1 - e^2) = r(1 + e \cos v) .$$

Differentiating Equations 7 with respect to time gives

$$\dot{x} = 2(\xi \dot{\xi} - \eta \dot{\eta}) , \quad \dot{y} = 2(\xi \dot{\eta} + \eta \dot{\xi}) .$$

Therefore

$$v^2 = \dot{x}^2 + \dot{y}^2 = 4(\xi^2 + \eta^2)(\dot{\xi}^2 + \dot{\eta}^2) = 4r(\dot{\xi}^2 + \dot{\eta}^2) .$$

Hence, the new expression of the square of the velocity in the  $\xi\eta$ -plane will be

$$\dot{\xi}^2 + \dot{\eta}^2 = \frac{\dot{x}^2 + \dot{y}^2}{4r} = \frac{v^2}{4r} ,$$

and, from the energy integral for Keplerian motion,

$$v^2 = \frac{2}{r} - \frac{1}{a} ,$$

it follows that

$$\dot{\xi}^2 + \dot{\eta}^2 = \frac{1}{2r^2} - \frac{1}{4ar} .$$

This expression approaches infinity as  $r$  approaches zero but, if the pseudo-time  $s$  is used as independent variable instead of  $t$ ,

$$\xi'^2 + \eta'^2 = \frac{1}{2} - \frac{r}{4a} . \quad (8)$$

In Levi-Civita coordinates, the velocity of impact is  $1/\sqrt{2}$ . The movement in the  $\xi\eta$ -plane is therefore regular and finite for all types of orbits and all phases of the motion. In the case of a parabolic orbit ( $a = \infty$ ), the velocity is constant  $= 1/\sqrt{2}$ .

The equations of motion in the plane,

$$\ddot{x} + \frac{x}{r^3} = 0 , \quad \ddot{y} + \frac{y}{r^3} = 0 ,$$

now take the form

$$\xi'' + \xi \left( \frac{1}{2r} - \frac{\xi'^2 + \eta'^2}{r} \right) = 0 ,$$

$$\eta'' + \eta \left( \frac{1}{2r} - \frac{\xi'^2 + \eta'^2}{r} \right) = 0 .$$

Equation 8 for  $\xi'^2 + \eta'^2$  gives:

$$\xi'' + \omega^2 \xi = 0 , \quad \eta'' + \omega^2 \eta = 0 , \quad \omega^2 = \frac{1}{4a} . \quad (9)$$

These differential equations are of the same type as before. As  $\xi'' + \omega^2 \xi = 0$ , etc., the general solution may be written

$$\xi = \xi_0 + c_1 s \xi_0' + c_2 s^2 \xi_0'' , \text{ etc.} \quad [c_i = c_i(\omega^2 s^2)] .$$

But, as  $\xi_0'' - \omega^2 \xi_0 = 0$ , therefore, with  $\lambda = \omega s$ ,

$$\xi = \xi_0 (1 - \lambda^2 c_2) + c_1 s \xi_0' ,$$

and, because  $1 - \lambda^2 c_2 = c_0$ ,

$$\xi = \xi_0 c_0 + \xi_0' c_1 s ,$$

$$\eta = \eta_0 c_0 + \eta_0' c_1 s ,$$

$$[c_i = c_i(\omega^2 s^2)] \quad (10)$$

or, in the elliptic case ( $\omega^2 > 0$ ),

$$\xi = \xi_0 \cos \omega s + \xi_0' s \frac{\sin \omega s}{\omega s} ,$$

$$\eta = \eta_0 \cos \omega s + \eta_0' s \frac{\sin \omega s}{\omega s} . \quad (11)$$

This solution is much simpler than the first one discussed.

The main equation is the same as before:

$$\tau = \int r ds = r_0 s + r_0' c_2 s^2 + r_0'' c_3 s^3, \quad c_i = c_i(\alpha^2 s^2).$$

But, as  $r_0'' = 1 - \alpha^2 r_0$ ,  $1 - c_3 \alpha^2 s^2 = c_1$ , and  $\alpha^2 = 4\omega^2$ :

$$\tau = r_0 c_1 s + r_0' c_2 s^2 + c_3 s^3.$$

Here the argument of the  $c$ -functions, however, is not  $\omega^2 s^2$  but

$$\alpha^2 s^2 = 4\omega^2 s^2.$$

The Levi-Civita transformation is only valid for plane coordinates. The next section shows that it is possible (though difficult) to extend this method to the case of three dimensions. Here, spinors prove a helpful device.

## LEVI-CIVITA TRANSFORMATIONS IN SPACE—INTRODUCTION OF SPINORS

The solutions of the plane two-body problem (given in Equations 10 and 11, expressed in Levi-Civita coordinates and with pseudo-time  $s$  as independent variable) prove to be simple, independent of the type of orbits, real, and finite even for collisions, and therefore easily computable even for small separations. The two-body problem lent itself to simplification through the use of the Levi-Civita transformations, as it could be restricted to two dimensions by the choosing of a convenient plane system of reference. The Levi-Civita transformations cannot be extended easily to three dimensions—a disadvantage in computing disturbed motions, as disturbing forces generally have components in all directions. Nevertheless, this extension can be made.

P. Kustaanheimo and E. Stiefel (References 2 and 3) have succeeded in constructing Levi-Civita coordinates in three dimensions by using spinors. A spinor is defined by an arrow in the complex  $z$ -plane, leading from a complex number  $z_1 = u_1 + i v_1$  to another  $z_2 = u_2 + i v_2$ . Consider such a spinor:

$$S_{12} = (z_1, z_2) = \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix}.$$

Then,

$$\bar{z}_1 = u_1 - i v_1$$

and

$$\bar{z}_2 = u_2 - i v_2 .$$

are the conjugates of  $z_1$  and  $z_2$ . Let us define four values  $r, x, y, z$  by

$$r + x = 2z_1 \bar{z}_1 = 2(u_1^2 + v_1^2) ,$$

$$r - x = 2z_2 \bar{z}_2 = 2(u_2^2 + v_2^2) ,$$

$$y + iz = 2z_1 \bar{z}_2 = 2(u_1 u_2 + v_1 v_2) + 2i(u_2 v_1 - u_1 v_2) ,$$

$$y - iz = 2z_2 \bar{z}_1 = 2(u_1 u_2 + v_1 v_2) - 2i(u_2 v_1 - u_1 v_2) ,$$

These equations give the relations

$$r = z_1 \bar{z}_1 + z_2 \bar{z}_2 = u_1^2 + v_1^2 + u_2^2 + v_2^2 ,$$

$$x = z_1 \bar{z}_1 - z_2 \bar{z}_2 = u_1^2 + v_1^2 - u_2^2 - v_2^2 ,$$

$$y = z_1 \bar{z}_2 + z_2 \bar{z}_1 = 2(u_1 u_2 + v_1 v_2) ,$$

$$z = i(z_2 \bar{z}_1 - z_1 \bar{z}_2) = 2(u_2 v_1 - u_1 v_2) . \quad (12)$$

These four values are not independent, though being functions of the four elements of the spinor  $S_{12}$ . Because

$$(r + x)(r - x) - (y + iz)(y - iz) = r^2 - (x^2 + y^2 + z^2) = 4(z_1 \bar{z}_1 \cdot z_2 \bar{z}_2 - z_1 \bar{z}_2 \cdot z_2 \bar{z}_1) = 0 .$$

Therefore

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{\sum (u_i^2 + v_i^2)}$$

is positive and equal to the absolute magnitude of the vector

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

From Equations 12, to every spinor  $S_{12} = (z_1, z_2)$  there corresponds a vector  $\vec{r}$ . This relation is not uniquely reversible. Indeed, if

$$z_1 = \rho_1 e^{i\psi_1}$$

and

$$z_2 = \rho_2 e^{i\psi_2},$$

Then, from Equation 12:

$$x = \rho_1^2 - \rho_2^2,$$

$$y = 2\rho_1 \rho_2 \cos(\psi_2 - \psi_1),$$

$$z = 2\rho_1 \rho_2 \sin(\psi_2 - \psi_1),$$

$$r = \rho_1^2 + \rho_2^2,$$

$$\psi_2 - \psi_1 = 2\phi. \quad (13)$$

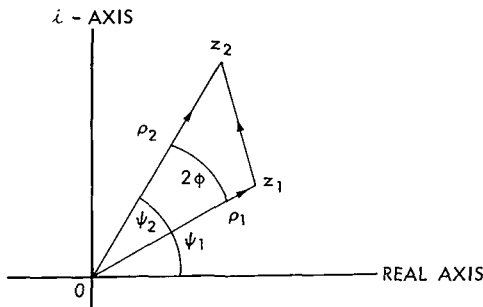


Figure 2—Representation of spinor  $Z_{12}$  in complex plane.

Therefore (see Figure 2), if we symbolize the spinor  $S_{12}$  by the arrow  $z_1 \rightarrow z_2$  in the complex plane or by the triangle  $\overrightarrow{Oz_1 z_2}$ , the vector  $\vec{r}$  is completely determined by the two positive numbers  $\rho_1, \rho_2$  and the angle  $2\phi$  between the directions  $Oz_1$  and  $Oz_2$ . That is, the vector  $\vec{r}$  is completely determined by the geometric form of triangle  $Oz_1 z_2$  and its sense of circulation, but is independent of the triangle's position in the plane, as given by angle  $\psi = (1/2)(\psi_1 + \psi_2)$ . Therefore,  $\psi$  is indeterminate and may be considered as an arbitrary function of time. Let  $S_{12}^*$  be one individual spinor selected from the multiplicity of spinors related to the vector  $\vec{r}$ . This multiplicity may be expressed by  $S_{12} = S_{12}^* e^{i\psi}$ , where  $\psi$  may denote any angle between 0 and  $2\pi$ .

We observe that the equations of transformation, Equations 12, go over to the Levi-Civita transformation of Equations 7, in the case of plane coordinates. Then  $z = 0$  and  $\psi_2 = \psi_1$  or  $\psi_1 + \pi$ . Thus, the two complex numbers are situated on a straight line through the zero-point of the complex plane, either on the same side or on opposite sides of the origin. The position angle of this line is  $\psi$  which, being arbitrary, may be set at zero. Then  $z_1$  and  $z_2$  lie on the real axis and are real



numbers  $\xi$  and  $\eta$ , such that

$$x = \xi^2 - \eta^2 ,$$

$$y = 2\xi\eta .$$

It is instructive to write down in spinor form some functions of the Cartesian coordinates and their derivatives, which are important in the theory of celestial motions. Such functions, in the case of undisturbed orbits, are:

the coordinates

$$x\dot{y} - y\dot{x} , \quad y\dot{z} - z\dot{y} , \quad z\dot{x} - x\dot{z}$$

of the constant vector  $\vec{r} \times \dot{\vec{r}}$  of the integral of areas,

the square of velocity,

$$\dot{\vec{r}}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \frac{2}{r} - \frac{1}{a} ,$$

which is an essential part of the integral of energy. Using Equations 12, the derivatives with respect to any independent variable  $s$  (denoting derivatives by primes), and the abbreviation

$$P = \bar{z}_1 z_1' - z_1 \bar{z}_1' + \bar{z}_2 z_2' - z_2 \bar{z}_2' , \quad (14)$$

gives

$$\begin{aligned} r' &= 2(z_1 \bar{z}_1' + z_2 \bar{z}_2') + P = 2(\bar{z}_1 z_1' + \bar{z}_2 z_2') - P , \\ x'^2 + y'^2 + z'^2 &= 4r(z_1 \bar{z}_1' + z_2 \bar{z}_2') + P^2 , \\ yz' - zy' &= 2i \left[ (z_1 \bar{z}_1' - \bar{z}_1 z_1') r + z_1 \bar{z}_1 P \right] , \\ zx' - xz' + i(xy' - yx') &= 2i \left[ (z_1 \bar{z}_2' - \bar{z}_2 z_1') r + z_1 \bar{z}_2 P \right] , \\ zx' - xz' - i(xy' - yx') &= 2i \left[ (z_2 \bar{z}_1' - \bar{z}_1 z_2') r + \bar{z}_1 z_2 P \right] . \end{aligned} \quad (15)$$

The last two of Equations 15 are each other's complex conjugates; replacing  $z_1, z_2, \bar{z}_1, \bar{z}_2, i$ , and  $P$  by their conjugates  $\bar{z}_1, \bar{z}_2, z_1, z_2, -i$  and  $\bar{P} (= -P)$  in either equation changes it into the other.

Since the vector  $\vec{r}$  in the  $xyz$ -space corresponds to a multiplicity of spinors  $S_{12}$ , this relation may be fixed by imposing a simplifying condition to be fulfilled by the elements of the spinor, namely that  $P = 0$ , so that Equations 15 become:

$$\begin{aligned}
 r' &= 2(z_1 \bar{z}_1' + z_2 \bar{z}_2') = 2(\bar{z}_1 z_1' + \bar{z}_2 z_2') , \\
 x'^2 + y'^2 + z'^2 &= 4r(z_1' \bar{z}_1' + z_2' \bar{z}_2') , \\
 yz' - zy' &= 2ir(z_1 \bar{z}_1' - \bar{z}_1 z_1') , \\
 zx' - xz' + i(xy' - yx') &= 2ir(z_1 \bar{z}_2' - \bar{z}_2 z_1') , \\
 zx' - xz' - i(xy' - yx') &= 2ir(z_2 \bar{z}_1' - \bar{z}_1 z_2') .
 \end{aligned} \tag{16}$$

But first we must prove that this does not affect the universality of vector  $\vec{r}$ , i.e., that the relation  $P = 0$  is independent of Equations 12 which define  $\vec{r}$  as a function of spinor  $S_{12}$ .

Since

$$z_1 = \rho_1 e^{i\psi_1}$$

and

$$z_2 = \rho_2 e^{i\psi_2} ,$$

then

$$P = 2i(\rho_1^2 \psi_1' + \rho_2^2 \psi_2') .$$

If  $P = 0$  and  $\psi_2 - \psi_1 = 2\phi$ , then

$$\begin{aligned}
 \rho_1^2 \psi_1' + \rho_2^2 \psi_2' &= 0 , \\
 \psi_1' - \psi_2' &= -2\phi' .
 \end{aligned} \tag{17}$$

From Equations (17):

$$\psi_1' = -\frac{2\rho_2^2}{\rho_1^2 + \rho_2^2} \phi' , \quad \psi_2' = +\frac{2\rho_1^2}{\rho_1^2 + \rho_2^2} \phi' .$$

If  $\psi = (1/2)(\psi_1 + \psi_2)$ , an angle by which the position of the spinor in the complex plane is fixed, then

$$\psi' = \frac{\rho_1^2 - \rho_2^2}{\rho_1^2 + \rho_2^2} \phi' = \frac{x}{r} \phi' ,$$

with arbitrary initial values  $\psi(0)$ —for instance  $\psi(0) = 0$ . Therefore, as  $\phi$  is a well-determined angle connected with the movement  $r(s)$ , the position angle  $\psi$  is fixed for every value of the independent variable  $s$ . The first of Equations 17 involves the statement that the movement of the spinor that corresponds to the condition  $P = 0$  may be interpreted as a motion of a system of two equal masses in the  $\rho, \psi$ -plane around the origin 0, the moment of impulses being constant and equal to zero.

The next step is to write in spinor form the differential equations for Kepler's Laws. We define the independent variable  $s$  by the differential relation

$$dt = r ds .$$

Transforming the differential equations

$$\ddot{x} + \frac{x}{r^3} = 0, \dots$$

directly into spinor equations is elementary but complicated. The transformation is simpler to realize starting from the known integrals of the two-body problem. These integrals give four algebraic relations which, by Equations 16, are:

$$y\dot{z} - z\dot{y} = \frac{1}{r} (yz' - zy') = 2i(z_1 \bar{z}_1' - \bar{z}_1 z_1') = \text{const.}$$

$$z\dot{x} - x\dot{z} + i(x\dot{y} - y\dot{x}) = \frac{1}{r} [zx' - xz' + i(xy' - yx')] = 2i(z_1 \bar{z}_2' - \bar{z}_2 z_1') = \text{const.}$$

$$z\dot{x} - x\dot{z} - i(x\dot{y} - y\dot{x}) = \frac{1}{r} [zx' - xz' - i(xy' - yx')] = 2i(z_2 \bar{z}_1' - \bar{z}_1 z_2') = \text{const.}$$

$$\left\{ \begin{array}{l} \dot{x}^2 + \dot{y}^2 + \dot{z}^2 - \frac{2}{r} = \frac{x'^2 + y'^2 + z'^2}{r^2} - \frac{2}{r} = -\frac{1}{a} = \text{const.} \\ \text{or } \frac{x'^2 + y'^2 + z'^2}{4r} - \frac{1}{2} = z_1' \bar{z}_1' + z_2' \bar{z}_2' - \frac{1}{2} = -\frac{r}{4a} . \end{array} \right. \quad (18)$$

Differentiating the first three relations of Equations 18 gives

$$z_1'' \bar{z}_1 - \bar{z}_1'' z_1 = 0 ,$$

$$z_1'' \bar{z}_2 - \bar{z}_2'' z_1 = 0 ,$$

$$z_2'' \bar{z}_1 - \bar{z}_1'' z_2 = 0 .$$

These equations let  $\bar{z}_2''$ ,  $\bar{z}_1''$ ,  $z_2''$ , be expressed in terms of  $z_1''$ :

$$\bar{z}_1'' = z_1'' \frac{\bar{z}_1}{z_1}, \quad \bar{z}_2'' = z_1'' \frac{\bar{z}_2}{z_1}, \quad z_2'' = z_1'' \frac{z_2}{z_1} . \quad (19)$$

The last of these equations can be written

$$z_1 z_2'' - z_2 z_1'' = 0 ,$$

delivering the spinor form of the integral of areas

$$z_1 z_2' - z_2 z_1' = \text{const.}$$

Differentiating the fourth of Equations 18 gives

$$z_1'' \bar{z}_1' + z_2'' \bar{z}_2' + \bar{z}_1'' z_1' + \bar{z}_2'' z_2' = -\frac{r'}{4a} .$$

Using Equations 19 to eliminate  $z_2''$ ,  $\bar{z}_1''$ ,  $\bar{z}_2''$  and multiplying by  $z_1$  gives

$$z_1'' (z_1 \bar{z}_1' + z_2 \bar{z}_2' + z_1' \bar{z}_1 + z_2' \bar{z}_2) = -z_1 \frac{r'}{4a} .$$

But, as the bracket equals  $r'$ ,

$$z_1'' + \frac{z_1}{4a} = 0 ,$$

and likewise (multiplying by  $z_2/z_1$ )

$$z_2'' + \frac{z_2}{4a} = 0 . \quad (20)$$

These are two linear second-order differential equations for the spinor  $S_{12} = (z_1, z_2)$ . Defining the spinor by its elements in vector form,

$$S_{12} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix},$$

gives four differential equations:

$$u_i'' + \omega^2 u_i = 0, \quad (i = 1, 2, 3, 4)$$

$$\omega^2 = \frac{1}{4a}, \quad (21)$$

which may be solved like the equations for  $\xi, \eta$  in the plane problem.

Solving these equations gives the corresponding solution for the Cartesian coordinates,  $x, y, z$  by Equations 12. However, the same problem expressed in space coordinates is perfectly determined by *six* independent constants, while Equations 21 require eight initial conditions,  $u_i(0), u_i'(0)$ . Two of the integral constants seem superfluous. But because of the relation  $P = 0$  between  $u_i$  and  $u_i'$ , only one of the eight initial conditions may be arbitrarily chosen between certain limits without influencing the character of the solution. Figure 2 makes this clearer. Triangle  $Oz_1 z_2$ , which represents the spinor, corresponds to a vector  $(x, y, z)$  only by its geometrical form and is independent of its position in the plane. The condition  $P = 0$  is identical with a first-order differential equation determining the rotational or oscillatory motion of the triangle around the zero-point of the complex plane, but this motion may begin with an arbitrary initial position.

Another question remains unanswered. In the equations

$$\xi'' + \omega^2 \xi = 0, \quad \eta'' + \omega^2 \eta = 0,$$

(which may stand now for the spinor equations of an undisturbed plane planetary motion), the integral constant of energy appears already in the equations themselves in the form of the constant  $\omega^2 = 1/a$ . This is confusing, indeed. However, these equations may be considered independent, as each of them is a differential equation in one variable ( $\xi$  or  $\eta$ ) only, with  $\omega^2$  as a parameter, which may be different for  $\xi$  and  $\eta$ , and is not a constant of integration. On the contrary,  $\omega^2$  represents the magnitude of the acting force causing an oscillatory movement of  $\xi$  or  $\eta$ . On the assumption that there are two different forces  $\omega_1^2$  for  $\xi$  and  $\omega_2^2$  for  $\eta$ , the motion of the  $\xi\eta$  system will be composed of two different oscillations with different periods for  $\xi$  and  $\eta$ , and this process cannot be considered as the Levi-Civita transformation of a motion in a Keplerian ellipse. This is only possible if the two parameters are identical and we consider the two equations as an inseparable system. Then

$\omega^2 = k^2 (m_1 + m_2) / 4a$  is likewise proportional to the acting gravitational force. The remarkable fact that the integration constant,  $a$ , appears connected with this parameter, while it does not appear in the differential equations for the Cartesian coordinates, is due to the transformation of time  $t$  into pseudo-time  $s$  by  $ds = dt/r$ , as  $a$  is the mean value of  $r$ .

## THE DIFFERENTIAL EQUATIONS OF THE SPINOR OF A PERTURBED KEPLER MOTION

The preceding sections have shown that for undisturbed planetary motion the introduction of spinors, together with a pseudo-time as independent variable, is advantageous. The spinor  $S(u_i)$ ,  $i = 1, 2, 3, 4$ , where the elements  $u_i$  satisfy the linear differential equations

$$u_i'' + \omega^2 u_i = 0, \quad (\omega^2 = \text{constant and real})$$

readily gives the Cartesian-coordinate solutions of the two-body problem from given  $u_i$ . The solutions  $u_i(s)$  are symmetric, simple, regular, and of the same form for all types of orbits. If the motion is in the  $xy$ -plane, we can do without spinors by using the Levi-Civita coordinates  $u_1, u_2$ , where

$$x = u_1^2 - u_2^2, \quad y = 2u_1 u_2.$$

Consider, now, the more complicated problem of planetary motion perturbed by one or more celestial bodies. Consideration of a single perturber is sufficient, as the transition to more than one is simple.

Let  $\vec{r}(t)$  be the heliocentric position vector of a planet. Then

$$\ddot{\vec{r}} = -\frac{\vec{r}}{r^3} + \ddot{\vec{p}} \quad (r = |\vec{r}|) \quad (22)$$

is the differential equation of its motion. The perturbation vector  $\ddot{\vec{p}}(t)$  may be decomposed into three components

$$\ddot{\vec{p}} = F\vec{r} + G\dot{\vec{r}} + H(\vec{r} \times \dot{\vec{r}}). \quad (23)$$

The components have the directions of the position vector  $\vec{r}$ , the velocity vector  $\dot{\vec{r}}$ , and a normal vector  $(\vec{r} \times \dot{\vec{r}})$  perpendicular to the plane determined by  $\vec{r}$  and  $\dot{\vec{r}}$ . The coefficients  $F, G, H$  are scalar functions of time  $t$  and of the heliocentric coordinates of the perturbed and perturbing bodies.

For the purpose of regularization we could define, as before, pseudo-time  $s$  by

$$ds = \frac{1}{r} dt. \quad (24)$$

But now we follow a proposal of P. Kustaanheimo (Reference 3), who more generally defines  $s$  by the relation

$$ds = \frac{A}{r} e^{-\int_0^t K dt} dt, \quad s = A \int_{t_0}^t \frac{1}{r} e^{-\int_0^t K dt} dt + B, \quad (25)$$

with  $A, B$  as arbitrary constants and  $K(t)$  as an arbitrary function of  $t$ . If  $A = 1, B = K = 0$ , Equation 25 changes into Equation 24.

Denoting derivatives with respect to  $s$  by primes, we have

$$\begin{aligned} t' &= \frac{1}{s} = \frac{r}{A} e^{\int_0^t K dt}, \\ t'' &= \frac{1}{A} e^{\int_0^t K dt} [r' + rKt'] = \frac{r' t'}{r} + Kt'^2. \end{aligned} \quad (26)$$

Let

$$\dot{\vec{r}} = \vec{r}' \dot{s} = \frac{\vec{r}'}{t'},$$

$$\ddot{\vec{r}} = \frac{\vec{r}''}{t'^2} - \frac{t'' \vec{r}'}{t'^3}.$$

Eliminating  $t''$  by Equation 26 and  $\ddot{\vec{r}}$  by Equations 22 and 23 gives

$$\begin{aligned} \vec{r}'' &= \left(F - \frac{1}{r^3}\right) \vec{r} t'^2 + \left[\frac{r'}{r} + (K+G) t'\right] \vec{r}' + H t' (\vec{r} \times \vec{r}'), \\ t'' &= \frac{r' t'}{r} + K t'^2. \end{aligned} \quad (27)$$

This is an 8th-order system composed of one vectorial and one scalar differential equation, both second-order. This system may be reduced to 6th-order by assigning special values to constants  $A$  and  $B$ . In the following,  $A = 1$  and  $B = 0$ .

Now let the elements of a spinor  $S = (z_1, z_2)$  replace the position coordinates  $x, y, z$  of the perturbed body. The two complex numbers  $z_j$  ( $j = 1, 2$ ) will be solutions of two differential equations, which are to be formulated. Spinor algebra permits the direct transformation of Equations 27 into spinor form (Reference 3). The spinor equation is shown to be identical with the first of Equations 27; this proves its validity.

Assume that the elements  $z_1$  and  $z_2$  of a spinor are solutions of the differential equations

$$z_j'' = \left( f t'^2 + \frac{P^2}{4r^2} \right) z_j + \left( g t' + \frac{P}{r} \right) \left[ z_j' - \frac{1}{2r} (r' + P) z_j \right] \\ (j = 1, 2). \quad (28)$$

Here

$$P = \bar{z}_1 z_1' - z_1 \bar{z}_1' + \bar{z}_2 z_2' - z_2 \bar{z}_2'.$$

is the function previously introduced in Equation 14;  $f$  and  $g$  are two arbitrary complex functions of  $s$ . As  $\bar{P} = -P$ , the equation for the complex conjugate  $\bar{z}_j$  is

$$\bar{z}_j'' = \left( \bar{f} t'^2 + \frac{P^2}{4r^2} \right) \bar{z}_j + \left( \bar{g} t' - \frac{P}{r} \right) \left[ \bar{z}_j' - \frac{1}{2r} (r' - P) \bar{z}_j \right], \\ (j = 1, 2). \quad (29)$$

Differentiating  $P$  gives

$$P' = \bar{z}_1 z_1'' - z_1 \bar{z}_1'' + \bar{z}_2 z_2'' - z_2 \bar{z}_2''.$$

Substituting the right-hand sides of Equations 28 and 29 for the second derivatives of the  $z$  and  $\bar{z}$  gives

$$P' = \left( f t'^2 + \frac{P^2}{4r^2} \right) (z_1 \bar{z}_1 + z_2 \bar{z}_2) + \left( g t' + \frac{P}{r} \right) \left[ z_1' \bar{z}_1 + z_2' \bar{z}_2 - \frac{r' + P}{2r} (z_1 \bar{z}_1 + z_2 \bar{z}_2) \right] \\ - \left( \bar{f} t'^2 + \frac{P^2}{4r^2} \right) (z_1 \bar{z}_1 + z_2 \bar{z}_2) - \left( \bar{g} t' - \frac{P}{r} \right) \left[ z_1 \bar{z}_1' + z_2 \bar{z}_2' - \frac{r' - P}{2r} (z_1 \bar{z}_1 + z_2 \bar{z}_2) \right].$$

From Equations 12 and 15,

$$r = z_1 \bar{z}_1 + z_2 \bar{z}_2, \quad r' + P = 2(z_1' \bar{z}_1 + z_2' \bar{z}_2), \quad r' - P = 2(z_1 \bar{z}_1' + z_2 \bar{z}_2').$$

Therefore,

$$P' = (f - \bar{f}) t'^2 r,$$



and if  $P = \text{const.} = 0$ , the function  $f$  must be real. In addition, let  $g = h + ik$ ,  $\bar{g} = h - ik$ , where  $h$  and  $k$  are real numbers. Then the differential Equations 28 and 29 take the simpler form

$$z_j'' = f t'^2 z_j + (h + ik) t' \left( z_j' - \frac{r'}{2r} z_j \right), \quad (30)$$

$$\bar{z}_j'' = f t'^2 \bar{z}_j + (h - ik) t' \left( \bar{z}_j' - \frac{r'}{2r} \bar{z}_j \right). \quad (31)$$

Equation 27 gives, for the  $x$ -component of the vector  $\bar{r}$ ,

$$x'' = \left( F - \frac{1}{r^3} \right) x t'^2 + \left[ \frac{r'}{r} + (K + G) t' \right] x' + H t' (y z' - z y'). \quad (32)$$

From  $x = z_1 \bar{z}_1 - z_2 \bar{z}_2$ :

$$x'' = z_1'' \bar{z}_1 + z_2'' \bar{z}_2' - z_2'' \bar{z}_2 - \bar{z}_2'' z_2 + 2(z_1' \bar{z}_1' - z_2' \bar{z}_2').$$

Eliminating the  $z''$  on the right side, using Equations 30 and 31, gives

$$\begin{aligned} x'' = & 2(z_1' \bar{z}_1' - z_2' \bar{z}_2') + 2(z_1 \bar{z}_1 - z_2 \bar{z}_2) \left( f t'^2 - h t' \frac{r'}{2r} \right) \\ & + h t' [\bar{z}_1 z_1' - \bar{z}_2 z_2' + z_1 \bar{z}_1' - z_2 \bar{z}_2'] + i k t' [\bar{z}_1 z_1' - \bar{z}_2 z_2' - z_1 \bar{z}_1' + z_2 \bar{z}_2']. \end{aligned}$$

From Equations 12 and 16:

$$x = z_1 \bar{z}_1 - z_2 \bar{z}_2,$$

$$x' = \bar{z}_1 z_1' - \bar{z}_2 z_2' + z_1 \bar{z}_1' - z_2 \bar{z}_2',$$

$$\frac{i}{r} (y z' - z y') = 2(\bar{z}_1 z_1' - z_1 \bar{z}_1') = \bar{z}_1 z_1' - \bar{z}_2 z_2' - z_1 \bar{z}_1' + z_2 \bar{z}_2'$$

as

$$\bar{z}_1 z_1' - z_1 \bar{z}_1' + \bar{z}_2 z_2' - z_2 \bar{z}_2' = P = 0.$$

Hence

$$\frac{r' x'}{r} - \frac{x}{2r^2} (x'^2 + y'^2 + z'^2) = 2(z_1' \bar{z}_1' - z_2' \bar{z}_2'),$$

and

$$\begin{aligned} x'' = x \left( 2ft'^2 - \frac{r'}{r} ht' - \frac{x'^2 + y'^2 + z'^2}{2r^2} \right) + x' \left( ht' + \frac{r'}{r} \right) \\ - (yz' - zy') \frac{kt'}{r} . \end{aligned} \quad (33)$$

Comparison of Equations 32 and 33 gives

$$\begin{aligned} 2ft'^2 - \frac{r'}{r} ht' - \frac{x'^2 + y'^2 + z'^2}{2r^2} &= \left( F - \frac{1}{r^3} \right) t'^2 , \\ ht' + \frac{r'}{r} &= \frac{r'}{r} + (K + G) t' , \\ - \frac{kt'}{r} &= Ht' . \end{aligned}$$

Equations 32 and 33 can be made identical by the following substitutions:

$$\left\{ \begin{aligned} k &= -rH , \\ h &= K + G , \\ 2f &= F - \frac{1}{r^3} + \frac{\dot{r}}{r} (K + G) + \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{2r^2} . \end{aligned} \right. \quad (34)$$

These conditions are independent of the position of the coordinate axes, and are also valid if we consider  $y$  or  $z$  instead of  $x$ . The energy-expression

$$E = \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{2} - \frac{1}{r} ,$$

which is constant for undisturbed orbits but slightly variable for disturbed ones, can be used to rewrite the third of Equations 34 as

$$2f = F + \frac{\dot{r}}{r} (K + G) + \frac{E}{r^2} .$$

K is still an arbitrary function, it may be equated with -G. Then

$$2f = F + \frac{E}{r^2}, \quad h = 0, \quad k = -rH,$$

and

$$ds = \frac{1}{r} e^{-\int G dt} dt.$$

Then the differential equations of the spinor are

$$z_j'' - ft'^2 z_j = ikt' \left( z_j' - \frac{r'}{2r} z_j \right),$$

or

$$z_j'' - \frac{Et'^2}{2r^2} z_j = \left( \frac{F}{2} t'^2 + \frac{i}{2} Ht' r' \right) z_j - irHt' z_j'.$$

As F and H are quantities of the order of the perturbing mass, the right sides of these equations are always small. The coefficient

$$\frac{Et'^2}{2r^2} = \frac{E}{2} e^{-2\int G dt}$$

is not very different from E/2, as G is of the order of the perturbations. E is not very different from -1/2a, if 2a is the length of the major axis of the osculating ellipse. Therefore let

$$z_j'' + \omega^2 z_j = Q_j, \quad (35)$$

where  $\omega^2$  is nearly constant, and positive in the elliptic case  $E < 0$ . The  $Q_j$  are expressions of the order of the perturbations.

The next section discusses the above results from another point of view. Equations 35 assume a form in which the coefficient of  $z_j$  on the left side is a true constant. Then these equations represent oscillations modulated by small external forces and well suited for numerical integration. They may be also the starting point of a new theory of general perturbations, which still has to be fully developed.

## VECTORIAL REPRESENTATION OF SPINORS

The previous section discussed the differential equations for a spinor  $(z_1, z_2)$  corresponding to the position vector  $(x, y, z)$  of a celestial body moving around the sun and subjected to perturbations by one of the major planets. Let  $F, G, H$  be the components of the disturbing acceleration their directions are those of the position vector  $\vec{r}$ , the velocity vector  $\dot{\vec{r}}$ , and the normal vector  $(\vec{r} \times \dot{\vec{r}})$ , respectively. Then the two complex elements  $z_1$  and  $z_2$  of the corresponding spinor are solutions of the differential equations

$$z_j'' = f t'^2 z_j + (h + ik) t' \left( z_j' - \frac{r'}{r} z_j \right) \quad (j = 1, 2). \quad (36)$$

Here the real quantities  $f, h, k$  are defined by

$$\begin{aligned} 2f &= F + \frac{E}{r^2} + \frac{\dot{r}}{r} (K + G), \\ h &= K + G, \\ k &= -rH. \end{aligned} \quad (37)$$

Also,

$$E = \frac{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}{2} - \frac{1}{r} \quad (38)$$

is an energy expression, which is slightly variable for perturbed motion, and is constant for undisturbed motion.  $K$  is an arbitrary function of time  $t$ , and can still be specified. The derivatives  $z', z'', t'$  are taken with respect to pseudo-time  $s$ , which is related to  $t$  by the differential equation

$$\frac{ds}{dt} = \frac{1}{r} e^{-\int_{t_0}^t K dt}, \quad (39)$$

with initial condition  $s = 0$  for  $t = t_0$ .

Giving the arbitrary function  $K(t)$  different forms will specialize these equations. Two possibilities for  $K(t)$  are of especial interest:  $K = -G$  and  $K = 0$ . As already shown, the first of these special assumptions leads to

$$2f = F + \frac{E}{r^2}, \quad h = 0, \quad k = -rH$$

and to the spinor equations

$$z_j'' - \frac{Et'^2}{2r^2} z_j = \frac{1}{2} (Ft'^2 + iHt' r') z_j - i r H t' z_j' . \quad (j = 1, 2)$$

These differential equations are of the type

$$z_j'' + \omega^2 z_j = Q_j ,$$

where  $\omega^2 = -Et'^2/2r^2$  is nearly constant, and the  $Q_j$  are complex functions of  $s$  and have magnitudes of the order of the perturbing forces.

Consider the special case where the perturbing planet moves in the same plane as the disturbed body. The problem becomes a plane one as the perturbation has no component perpendicular to the plane of movement, and the  $z_j$  are real and identical with the Levi-Civita coordinates. Therefore,  $H = k = 0$ , and

$$z_j'' - ft'^2 z_j = 0 . \quad (j = 1, 2) .$$

This system of equations has the integral of areas  $z_1 z_2' - z_2 z_1' = \text{const.}$ , as in the case of undisturbed orbits. This is a remarkable result. In its classical form, the problem of a small body ( $m = 0$ ) moving in the plane of the orbit of a perturbing planet (Jupiter) generally has no integral of motion of simple algebraic form. With the conventional treatment, the useful integral of Jacobi is only available in the *restricted* three-body problem—for a circular Jupiter-orbit. Yet here after the introduction of spinors (i.e., Levi-Civita coordinates), such an integral exists *always*, without any reservation. Naturally, the existence of this integral is not enough to solve the problem, but it helps to indicate a solution.

Consider the second special proposal for the function  $K(t)$ ; i.e.,  $K = 0$ . Then

$$\frac{ds}{dt} = \frac{1}{r} , \quad s(0) = 0$$

defines the pseudo-time  $s$  in a new form. Since the work done by Sundman, Levi-Civita, Stumpff, and others, this form has often been accepted for a regularizing variable in celestial mechanics. We have

$$\begin{aligned} t' &= r , & h &= G , & k &= -rH , & \dot{\mathbf{x}} &= \frac{\mathbf{x}'}{r} , \dots \\ 2ft'^2 &= r^2 F + r r' G + E = r^2 F + r' G + E , \\ E &= \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{r} = \frac{x'^2 + y'^2 + z'^2}{2r^2} - \frac{1}{r} , \end{aligned}$$

and the spinor equations ( $j = 1, 2$ )

$$z_j'' + z_j \left( \frac{1}{2r} - \frac{x'^2 + y'^2 + z'^2}{4r^2} \right) = \frac{1}{2} z_j (r^2 F + i r r' H) + r z_j' (G - i r H). \quad (40)$$

Equation 40 represents four equations determining the four real quantities  $u_1 \cdots u_4$  that compose the spinor

$$S = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} u_1 + i u_2 \\ u_3 + i u_4 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}.$$

The spinor  $S$  may be replaced by the vector  $(u_i)$  in the 4-dimensional space  $R_4$ . This is not as illustrative as its definition by a pair of complex numbers, but it is more uniform and permits the elimination of the imaginary  $i$  from Equation 40.

The vectorial form of spinors has many advantages. Consider first the case of plane motion. The Levi-Civita transformation,

$$x_1 = u_1^2 - u_2^2, \quad x_2 = 2u_1 u_2, \quad (41)$$

maps the two-dimensional vector  $(x_1, x_2)$  onto the two-dimensional  $(u_1, u_2)$  plane. Differentiating gives

$$dx_1 = 2(u_1 du_1 - u_2 du_2),$$

$$dx_2 = 2(u_2 du_1 + u_1 du_2),$$

or, in vectorial form,

$$\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix} = 2M \begin{pmatrix} du_1 \\ du_2 \end{pmatrix} \quad (42)$$

with the transformation matrix

$$M = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix}.$$

$M$  is orthogonal and has the norm  $u_1^2 + u_2^2$ . Integrating Equation 42 gives

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = M \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

From vector and matrix algebra,

$$r = \sqrt{\sum x_i^2} = \sum u_i^2,$$

$$v^2 = \sum x_i'^2 = 4 \sum u_i^2 \sum u_i'^2 = 4r \sum u_i'^2.$$

Vectors in space present difficulties in mapping  $R_3$  onto the four-dimensional  $R_4$ . According to Hurwitz, transformations of the form in Equation 42 can be made only for spaces of dimensionality 1, 2, 4 and 8. Therefore,

$$\begin{pmatrix} x_1 \\ \vdots \\ x_4 \end{pmatrix} = M \begin{pmatrix} u_1 \\ \vdots \\ u_4 \end{pmatrix},$$

where

$$M = \begin{pmatrix} u_1 - u_2 - u_3 & u_4 \\ u_2 & u_1 - u_4 - u_3 \\ u_3 & u_4 & u_1 & u_2 \\ u_4 - u_3 & u_2 - u_1 \end{pmatrix}$$

is orthogonal and has the norm  $\sum u_i^2$ . This leads to

$$(dx_i) = 2M(du_i),$$

$$\sqrt{\sum x_i^2} = r = \sum u_i^2, \tag{43}$$

$$\sum x_i'^2 = 4r \sum u_i'^2.$$

These relations correspond exactly to those of two dimensions. Explicitly:

$$x_1 = u_1^2 - u_2^2 - u_3^2 + u_4^2 ,$$

$$x_2 = 2(u_1 u_2 - u_3 u_4) ,$$

$$x_3 = 2(u_1 u_3 + u_2 u_4) ,$$

$$x_4 = 0$$

and

$$dx_1 = 2(u_1 du_1 - u_2 du_2 - u_3 du_3 + u_4 du_4) ,$$

$$dx_2 = 2(u_2 du_1 + u_1 du_2 - u_4 du_3 - u_3 du_4) ,$$

$$dx_3 = 2(u_3 du_1 + u_4 du_2 + u_1 du_3 + u_2 du_4) ,$$

$$dx_4 = 2(u_4 du_1 - u_3 du_2 + u_2 du_3 - u_1 du_4) = 0 .$$

The expression on the right side of the last equation is not a complete differential. Equating it to zero gives the same relation, namely  $P = 0$ —previously assumed as an accessory condition to make the relation between vector and spinor unique. Here, this condition emerges naturally. We can map the three-dimensional  $x_i$ -space onto the four-dimensional  $u_i$ -space in the following manner: from the four-dimensional  $x_i$ -space take out the three-dimensional section  $x_4 = 0$ ; then the corresponding  $u_i$ -space will be specialized by the differential condition  $P' = 0$ .

We can transform the differential equations in  $x_i$  directly into those in the corresponding  $u_i$ -coordinates—for instance, by transforming the forces  $p_i$  ( $x_1 \cdots x_3$ ) into corresponding forces  $S_i$  ( $u_1 \cdots u_4$ ), where the work done by the forces must be independent of the coordinate system, i.e.

$$\sum_{i=1}^3 p_i dx_i = \sum_{i=1}^4 S_i du_i .$$

A most interesting study based on the above method is presented by Kustaanheimo and Stiefel (Reference 2).

Considering Equations 43, write Equation 40 in the form

$$z_j'' + \frac{1}{r} z_j \left( \frac{1}{2} - \sum u_i'^2 \right) = \frac{1}{2} z_j (Fr^2 + ir r' H) + z_j' r(G - irH) = rR_j .$$



Let

$$R_1 = S_1 + iS_2, \quad R_2 = S_3 + iS_4.$$

Then the spinor equation can be rewritten as four equations involving only real quantities

$$u_i'' + \frac{1}{r} \left( \frac{1}{2} - \sum u_j'^2 \right) u_i = r S_i, \quad (i = 1, 2, 3, 4) \quad (44)$$

where

$$\begin{aligned} r &= \sum u_j^2, & r' &= 2 \sum u_j u_j', \\ S_1(u_1, u_2) &= \frac{1}{2} u_1 Fr - \frac{1}{2} u_2 r'H + u_1'G + u_2' rH, \\ S_2(u_1, u_2) &= \frac{1}{2} u_2 Fr + \frac{1}{2} u_1 r'H + u_2'G - u_1' rH, \\ S_3 &= S_1(u_3, u_4), \\ S_4 &= S_2(u_3, u_4). \end{aligned}$$

These equations have the form of Equation 35:

$$u_i'' + \omega^2 u_i = r S_i,$$

where  $\omega^2$  is nearly constant. Instead of  $s$ , introduce another independent variable  $q$  such that Equations 44 take the form

$$\frac{d^2 u_i}{dq^2} + \omega^2 u_i = Q_i, \quad (i = 1, 2, 3, 4)$$

with constant  $\omega^2$ . This is the case of simple harmonic oscillation modulated by small external forces  $Q_i$ . Indicating derivatives with respect to  $q$  by bars gives

$$\bar{u} = \frac{du}{dq}, \quad \bar{s} = \frac{ds}{dq}.$$

Then

$$u' = \frac{du}{ds} = \frac{\bar{u}}{\bar{s}}, \quad u'' = \frac{d^2 u}{ds^2} = \frac{\bar{\bar{u}}}{\bar{s}^2} - \frac{\bar{u}\bar{\bar{s}}}{\bar{s}^3},$$

Then, instead of Equation 44,

$$\bar{\bar{u}}_i + \frac{u_i}{r} \left( \frac{\bar{s}^2}{2} - \sum \bar{u}_j^2 \right) = r \bar{s}^2 S_i + \bar{u}_i \frac{\bar{\bar{s}}}{\bar{s}} . \quad (45)$$

If

$$\omega^2 = \frac{1}{r} \left( \frac{\bar{s}^2}{2} - \sum \bar{u}_j^2 \right)$$

is a constant, then,

$$\bar{s}^2 = 2 \left( r \omega^2 + \sum \bar{u}_j^2 \right) , \quad (46)$$

and Equation 45 gives

$$\bar{\bar{u}}_i + \omega^2 u_i = 2 r S_i \left( r \omega^2 + \sum \bar{u}_j^2 \right) + \bar{u}_i \frac{\bar{\bar{s}}}{\bar{s}} .$$

From Equation 46, with constant  $\omega^2$ ,

$$\begin{aligned} \bar{s} \bar{\bar{s}} &= \bar{r} \omega^2 + 2 \sum \bar{u}_j \bar{\bar{u}}_j \\ &= 2 \sum \bar{u}_j \left( \bar{\bar{u}}_j + \omega^2 u_j \right) . \end{aligned}$$

Introducing Equation 45 gives, finally,

$$\frac{1}{2} \bar{s} \bar{\bar{s}} = r \bar{s}^2 \sum S_j \bar{u}_j + \frac{\bar{\bar{s}}}{\bar{s}} \sum \bar{u}_j^2 . \quad (47)$$

This may be written

$$\frac{\bar{\bar{s}}}{\bar{s}} \left( \frac{1}{2} \bar{s}^2 - \sum \bar{u}_j^2 \right) = r \bar{s}^2 \sum S_j \bar{u}_j ,$$

which, with the aid of Equation 46, becomes

$$\frac{\bar{\bar{s}}}{\bar{s}} r \omega^2 = r \bar{s}^2 \sum S_j \bar{u}_j .$$

Therefore, instead of Equation 45:

$$\begin{aligned}\bar{u}_i + \omega^2 u_i &= r S_i \bar{s}^2 + \bar{u}_i \frac{\bar{s}^2}{\omega^2} \sum S_j \bar{u}_j \\ &= 2(r\omega^2 + \sum \bar{u}_j^2) \left( r S_i + \frac{\bar{u}_i}{\omega^2} \sum S_j \bar{u}_j \right) \equiv Q_i\end{aligned}\quad (48)$$

This is a system of equations of the required form. To regain the time  $t$  from the pseudo-time  $q$ , put

$$\frac{dt}{dq} = \frac{dt}{ds} \cdot \frac{ds}{dq} = r \sqrt{2(r\omega^2 + \sum \bar{u}_j^2)}$$

The square root is not very different from unity. For undisturbed motion,  $\omega^2 = 1/4a$ , and

$$\sum \bar{u}_j^2 = \sum \dot{u}_j^2 \left( \frac{dt}{dq} \right)^2 = \frac{V^2}{4r} \cdot 2r^2 \left( \frac{r}{4a} + \sum \bar{u}_j^2 \right)$$

with

$$V^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \frac{2}{r} - \frac{1}{a}.$$

From this relation:

$$\sum \bar{u}_j^2 \left( 1 - \frac{V^2 r}{2} \right) = \sum \bar{u}_j^2 \cdot \frac{r}{2a} = \frac{V^2 r}{8a} = \frac{r}{4a} \left( 1 - \frac{r}{2a} \right);$$

i.e.,

$$\sum \bar{u}_j^2 = \frac{1}{2} - \frac{r}{4a}.$$

With  $r\omega^2 = r/4a$ , it follows that, for undisturbed motion,

$$\frac{ds}{dq} = \sqrt{2(r\omega^2 + \sum \bar{u}_j^2)} = 1.$$

As the constant  $\omega^2$  can be arbitrarily chosen, it will always be suitable to put  $\omega^2 = 1/4a$ , if  $a$  is the semi-major axis of an osculating or intermediary Keplerian ellipse.

This method of using spinors in their vectorial form to solve the problem of computing disturbed orbits has been successfully applied to minor planets at the Institute of Applied Mathematics at Zürich, Switzerland (Reference 4). Difficulties arise in the case of orbits with parabolic or near-parabolic character. Then  $\omega^2 = 0$  and on the right side of Equations 48  $\omega^2$  is in the denominator. This drawback disappears only if  $\Sigma S_j \bar{u}_j$  goes to zero with  $\omega^2$ . The vanishing of  $\Sigma S_j \bar{u}_j$  occurs for all cases of undisturbed motion, as  $\bar{s} = 1$ , and therefore  $\bar{\bar{s}} = 0$ . But in general, if  $\omega = 0$ , Equation 46 shows that  $\bar{s} = 1$  only when  $\Sigma \bar{u}_j^2 = \text{const.} = 1/2$ . This agrees with what was said about Equation 8. Thus, with spinor coordinates and with pseudo-time  $q$  instead of  $t$ , the expression for the generalized kinetic energy is constant in the parabolic case; therefore, the work done by the acting forces (defined as the variation of the kinetic energy with time)

$$\Sigma S_j \bar{u}_j = 0 \quad .$$

The above drawback is not critical; there are many ways to overcome it. For example,  $\Sigma S_j \bar{u}_j$  can be developed in a series

$$\Sigma S_j u_j = \alpha_1 \omega^2 + \alpha_2 \omega^4 + \alpha_3 \omega^6 + \dots \quad ,$$

that converges rapidly for small  $\omega^2$  and has  $\omega^2$  as a factor.

The next section discusses some ideas about regularizing dynamical processes in general. The difficulties with highly eccentric orbits are found avoidable from the beginning.

## GENERAL CONSIDERATIONS ON REGULARIZATION IN DYNAMICAL PROBLEMS

The preceding section presented a survey of the use of spinors in celestial mechanics. Introducing these symbols, together with a certain pseudo-time as independent variable, made the equations of celestial movements become simple, linear, and with regular solutions even in the case of collisions; which would not occur with Cartesian space coordinates and time as variables. Before going more deeply into the questions of linearization and regularization, let us review the useful attributes of spinors.

Spinors are most obviously useful for analyzing undisturbed motions of a body around its central mass point. The equations of motion of the body with respect to the central mass are

$$\ddot{x} + \frac{x}{r^3} = 0 \quad , \quad \ddot{y} + \frac{y}{r^3} = 0 \quad , \quad \ddot{z} + \frac{z}{r^3} = 0 \quad ,$$

if the units of time and mass are so chosen that the constant expression  $k^2(m_1 + m_2)$  is equal to unity and therefore disappears from the equations. The distance  $r$  satisfies the second-order

differential equation

$$r \ddot{r} + \dot{r}^2 - \frac{1}{r} = -\alpha^2 ,$$

where  $\alpha^2$  is a constant equal to  $1/a$ ,  $a$  being the major semi-axis of the conic section orbit described by the body. These equations can be transformed into a linear form if pseudo-time  $s$  instead of time  $t$  is used as independent variable, such that

$$dt = r ds , \quad (s = 0 \quad \text{for} \quad t = t_0) .$$

Then the differential equations for  $r$  and  $x$  take the form

$$r'' + \alpha^2 r = 1 , \quad x'' + \alpha^2 x = \frac{1}{r} (x' r' - x r'') = \text{const.}$$

The equations for  $q = r, x, y, z$  respectively, have the common form

$$q''' + \alpha^2 q' = 0 , \quad \alpha^2 = \frac{1 - r''}{r} = \frac{1 - r_0''}{r_0} , \quad q' = \frac{dq}{ds} . \quad (49)$$

The solutions of these linear differential equations may be written

$$q(s) = q_0 + c_1 s q_0' + c_2 s q_0'' . \quad (50)$$

Here  $q_0, q_0', q_0''$  are the values of  $q, q', q''$  at the initial time  $t = t_0$ , and

$$c_1(\lambda^2) , \quad c_2(\lambda^2) , \quad \dots , \quad \lambda = \alpha s ,$$

belong to the series of  $c$ -functions

$$c_0 = \cos \lambda , \quad c_1 = \frac{\sin \lambda}{\lambda} , \quad c_2 = \frac{1 - \cos \lambda}{\lambda^2} , \quad c_3 = \frac{\lambda - \sin \lambda}{\lambda^2} , \quad \dots , \quad (51)$$

which are derived from the generating function  $c_0 = \cos \lambda$  by

$$c_n = \frac{1}{s^n} \int_0^s \dots \int_0^s \cos \lambda (ds)^n .$$

In particular,

$$r(s) = r_0 + c_1 s r_0' + c_2 s^2 r_0'' ,$$

and the connection between intermediate time  $\tau = t - t_0$  and pseudo-time  $s$  is given by the "main equation"

$$\tau = \int_0^s r(s) ds = \int_0^s (r_0 + c_1 s r_0' + c_2 s^2 r_0'') ds ;$$

i.e.,

$$\tau = s r_0 + c_2 s^2 r_0' + c_3 s^3 r_0'' ,$$

if we use the integral relations for the  $c$ -functions,

$$\int_0^s c_n s^n ds = c_{n+1} s^{n+1} .$$

Equation 50 (the solution of Equation 49) is regular with respect to pseudo-time  $s$ . Indeed, in the case of rectilinear motion the velocity  $V(s) = \sqrt{\dot{x}'^2 + \dot{y}'^2 + \dot{z}'^2}$  is zero at the time of impact, while  $V(\tau) = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$  approaches infinity. On the other hand, there remains a certain singularity of position. For when the falling body reaches the central mass point, the pseudo-velocity goes to zero and the direction of motion reverses. The space trajectory therefore has a cusp at this moment. In any rotating system the trajectory will appear as an ejection orbit. This singularity is overcome by using the elements  $u_i$  of a spinor instead of the Euclidean coordinates, for then:

1. The differential equations take the form of Equation 21:

$$u_i'' + \omega^2 u_i = 0 , \quad \omega^2 = \frac{1}{4a} = -\frac{1}{4} \alpha^2 ,$$

and are satisfied by general solutions analogous to Equations 10:

$$u_i = c_0 u_{i0} + c_1 s u_{i0}' , \quad c_n = c_n (\omega^2 s^2) ,$$

which are noticeably simpler than those of the ordinary rectangular coordinates.

2. The singularity of position in the case of impact disappears. In terms of the Levi-Civita coordinates  $\xi, \eta$ , the body passes the point of impact in a straight line with finite constant velocity  $\neq 0$ .

This theory, as proposed by Kustaanheimo and Stiefel, was then extended to the more complicated case of perturbed orbits. The perturbed differential equations of the spinor elements  $u_i$  take the form

$$u_i'' + \omega^2 u_i = Q_i(s) , \quad (52)$$

the  $Q_i(s)$  being small quantities of the order of the perturbing masses. The "frequencies"  $\omega^2$  are no longer constant, but are slightly variable functions of  $s$ . To avoid this inconvenience, we transformed Equation 52 once more so that  $\omega^2$  became constant. This was made possible by introducing instead of  $s$  a new pseudo-time  $q$ . If derivatives with respect to  $q$  are indicated by bars,

$$\frac{du}{dq} = \bar{u} ,$$

the differential equations take the form

$$\bar{\bar{u}}_i + \omega^2 u_i = S_i(q) ,$$

the  $S_i(q)$  being small quantities of the order of the perturbations, and the frequency  $\omega$  now being constant. Parabolic orbits present some difficulties. The  $S_i$  contain a term of the form  $\sigma/\omega^2$  which, in the parabolic case,  $\omega^2 = 0$ , seems to have the indeterminate value  $0/0$ , but approaches a finite-limit value as  $\omega$  approaches zero; it may be developed for small values of  $\omega^2$  into a rapidly converging power series. But this would necessitate the use of different computing methods for different types of orbits. This drawback is due to the introduction of  $q$  instead of  $s$ ; fortunately, there is another way of making the coefficient of  $u_i$  constant, without changing the variables again (see Reference 6, page 573 ff.).

Consider the perturbed differential equation of  $r(s)$ ,

$$r''' + \alpha^2 r' = R(s) = rf(s) , \quad \alpha^2 = \frac{1 - r''}{r} ,$$

where  $f(s)$  is a well-known function of the coordinates of the bodies that participate in the motion. Then

$$\frac{d\alpha^2}{ds} = -\frac{1}{r} (r''' + \alpha^2 r') = -f(s)$$

is not zero as it would be in the undisturbed case,  $f = 0$ . Integrating this equation gives

$$\alpha^2 = \alpha_0^2 - \int_0^s f(s) ds ,$$

where  $\alpha_0^2$  is a real constant of integration. Therefore,

$$r''' + \alpha_0^2 r' = rf(s) + r' \int_0^s f ds = \frac{d}{ds} \left( r \int_0^s f ds \right) = h'(s)$$

or, after integration,

$$r'' + \alpha_0^2 r = h(s) = 1 + k(s) ,$$

where  $k(s)$  is of the order of the perturbing masses and can be developed into a power series or a trigonometric-function series. Using the  $c$ -functions permits the special solutions of this equation to take the form

$$r(s) = r_0 + c_1 s r_0' + c_2 s^2 r_0'' + (c_3 s^3 k_0' + c_4 s^4 k_0'' + \dots) ,$$

the terms in brackets being small, and of the order of the perturbations. Similar solutions can be found for the rectangular coordinates  $x, y, z$  as functions of  $s$ , and likewise for the oblique-angled coordinates  $F, G, H$  defined by the relation

$$r = F\vec{r}_0 + G\dot{\vec{r}}_0 + H(\vec{r}_0 \times \dot{\vec{r}}_0) , \quad (53)$$

where the directions are those of  $\vec{r}_0$ , the position vector at  $t = t_0$ , the velocity vector  $\dot{\vec{r}}_0$ , and the vector perpendicular to the plane of  $\vec{r}_0$  and  $\dot{\vec{r}}_0$ , respectively. (Functions  $F, G, H$  in Equation 53 are different from those in Equation 23, which are the components of the acceleration  $\ddot{\vec{p}}$  of the perturbing vector.) The intermediate time  $\tau = t - t_0$  can be calculated as a function of  $s$  from

$$\tau = \int_0^s r ds = s r_0 + c_2 s^2 r_0' + c_3 s^3 r_0'' + (c_4 s^4 k r_0' + c_5 s^5 k_0'' + \dots) ,$$

which is the main equation of the problem.

This elegant solution of the special problem of perturbed planetary motions can easily be applied to the spinor equations of perturbed motion. In fact, the method can be extended to any



general dynamical problem whose differential equations can be transformed into linear differential equations by introducing new dependent and independent variables; or, the problem may have this form from the beginning (see Reference 7). Assume that a dynamical process  $p(s)$  may be described by the equation

$$p^{(n)} + a_1 p^{(n-1)} + a_2 p^{(n-2)} + \cdots + a_{n-1} p' + a_n p = 0, \quad (54)$$

the  $a_i$  being constants and  $s$  being either time or any pseudo-time. Then, if  $p_1, p_2, \cdots, p_n$  are  $n$  independent, simple, particular solutions of the problem, a generating linear function of the  $p_i$  may be written:

$$\phi(s) = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n,$$

the coefficients  $\alpha_i$  being constant or zero. From this generating function, a system of auxiliary functions  $\phi_1(s), \phi_2(s), \cdots, \phi_\nu(s), \cdots$  may be derived by putting

$$\phi_\nu(s) = \frac{1}{s^\nu} \int_0^s \cdots \int_0^s \phi(s) (ds)^\nu, \quad \nu = 1, 2, \cdots \quad (55)$$

As  $\phi$  is also a particular solution of Equation 54, it satisfies the equation

$$\phi^{(n)} + a_1 \phi^{(n-1)} + a_2 \phi^{(n-2)} + \cdots + a_{n-1} \phi' + a_n \phi = 0. \quad (56)$$

Any special solution of this equation may be given by  $n$  initial values for  $t = t_0$ ,

$$\phi_0, \phi_0', \phi_0'', \cdots, \phi_0^{(n-1)}.$$

Determine these values so that

$$\begin{aligned} \phi_0 &= 1, \\ \phi_0' + a_1 \phi_0 &= 0, \\ \phi_0'' + a_1 \phi_0' + a_2 \phi_0 &= 0, \\ &\cdots \cdots \cdots \\ \phi_0^{(n-1)} + a_1 \phi_0^{(n-2)} + \cdots + a_{n-1} \phi_0 &= 0. \end{aligned} \quad (57)$$

Then, after consecutive integration of Equation 56,

[illegible]

The last of Equations 58 can be written (if the integrals are eliminated by Equation 55):

$$\phi + a_1 s \phi_1 + a_2 s^2 \phi_2 + \dots + a_n s^n \phi_n = 1 \quad .$$

Integrating  $v$  times between 0 and  $s$ , and considering the initial conditions, Equations 57, gives

$$s^{\nu} \phi_{\nu} + a_1 s^{\nu+1} \phi_{\nu+1} + a_2 s^{\nu+2} \phi_{\nu+2} + \dots + a_n s^{\nu+n} \phi_{\nu+n} = \frac{s^{\nu}}{\nu!}, \quad (1)$$

or, dividing by  $s^\nu$ ,

$$\phi_\nu + a_1 s \phi_{\nu+1} + a_2 s^2 \phi_{\nu+2} + \cdots + a_n s^n \phi_{\nu+n} = \frac{1}{\nu!}. \quad (59)$$

Substituting the left side of Equation 59 for each  $1/\nu!$ -type factor in the Taylor series:

$$p(s) = p_0 + \frac{1}{1!} p_0' s + \frac{1}{2!} p_0'' s^2 + \frac{1}{3!} p_0''' s^3 + \dots$$

reduces this infinite power series to the finite form

$$p(s) = p_0 + s\phi_1 p_0' + s^2\phi_2(p_0'' + a_1 p_0') + \dots \\ + s^n \phi_n(p_0^{(n)} + a_1 p_0^{(n-1)} + \dots + a_{n-1} p_0') ,$$

because higher-order terms on the right side are zero.

This method can be shown to apply to many dynamical problems, for instance, the three following examples.

### 1. Equation

$$p''' + \alpha^2 p' = 0 , \quad (a_1 = 0, a_2 = \alpha^2, a_3 = 0)$$

of the two-body problem, the generating function being  $\phi = \cos \alpha s$ , which fulfills the conditions

$$\phi_0 = 1 , \quad \phi_0' + a_1 \phi_0 = 0 , \quad \phi_0'' + a_1 \phi_0' + a_2 \phi_0 = 0$$

and the recurrence formula

$$\phi_\nu + \alpha^2 s^2 \phi_{\nu+2} = \frac{1}{\nu!} .$$

The  $\phi_\nu$  are identical with the c-functions.

### 2. The differential equation of a periodic oscillation with two frequencies,

$$p = a_1 \cos(\alpha_1 s + A_1) + a_2 \cos(\alpha_2 s + A_2) .$$

This function is the solution of the linear and homogeneous differential equation

$$p^{(5)} + (\alpha_1^2 + \alpha_2^2) p^{(3)} + \alpha_1^2 \alpha_2^2 p' = 0 .$$

The auxiliary functions  $\phi(s)$  are derived from the generating function

$$\phi = \frac{\alpha_1^2}{\alpha_1^2 - \alpha_2^2} \cos \alpha_1 s - \frac{\alpha_2^2}{\alpha_1^2 - \alpha_2^2} \cos \alpha_2 s = 1 - (\alpha_1^2 + \alpha_2^2) \frac{s^2}{2!} + (\alpha_1^4 + \alpha_1^2 \alpha_2^2 + \alpha_2^4) \frac{s^4}{4!} - \dots ,$$

which obeys the conditions

$$\phi_0 = 1 ,$$

$$\phi_0' + a_1 \phi_0 = 0 ,$$

$$\dots\dots\dots$$

$$\dots\dots\dots$$

$$\phi_0^{(4)} + a_1 \phi_0''' + a_2 \phi_0'' + a_3 \phi_0' + a_4 = 0 ,$$

$$\phi_0' = 0 ,$$

$$\phi_0'' = - (a_1^2 + a_2^2) ,$$

$$\phi_0''' = 0 ,$$

$$\phi_0^{(4)} = a_1^4 + a_1^2 a_2^2 + a_2^4 .$$

### 3. The differential equation of a damped vibration,

$$p''' + 2kp'' + (k^2 + \alpha^2) p' = 0 .$$

In this case the generating function is

$$\phi = e^{-ks} \left( \cos \alpha s - \frac{k}{\alpha} \sin \alpha s \right) ,$$

from which the auxiliary functions

$$\phi_\nu = \frac{1}{s^\nu} \int_0^s \dots \int_0^s \phi(ds)^\nu$$

can be derived. Then

$$\frac{1}{\nu!} = \phi_\nu + 2ks \phi_{\nu+1} + (k^2 + \alpha^2) s^2 \phi_{\nu+2}$$

eliminates the reciprocal factorials from the Taylor development of  $p(s)$  and produces a closed formula. In these last two examples  $s$  is the time  $t - t_0$  itself, but we can imagine cases in which the simple form  $p(s)$  appears only after a convenient transformation of the coordinates.

The above considerations help to solve the two-body problem and may apply to more complicated problems. They have not been applied so far, but it would be very well worth while if they were. This method may be qualified for finding, for instance, closed mathematical expressions for certain periodic orbits in the restricted three-body problem.

The author of this paper has applied a similar method to solve Hill's problem of the periodic inequality of the motion of the moon or "variation" (Reference 6, pages 578-588). There are other

ways of using these ideas. For instance, the calculation of intermediate orbits of the planets or the moon; this includes calculating the osculating Keplerian orbit and some of the more important periodic terms of the perturbations. And doubtless there are many more.

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